

# On ‘Arnold’s theorem’ on the stability of the solar system

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Consider  $1 + n$  point bodies with masses  $m_0, \epsilon m_1, \dots, \epsilon m_n > 0$  ( $\epsilon > 0$  and  $n \geq 2$ ) and position vectors  $x_0, x_1, \dots, x_n \in \mathbb{R}^3$ , undergoing Newton’s universal attraction:

$$\ddot{x}_0 = \epsilon \sum_{k \geq 1} m_k \frac{x_k - x_0}{\|x_k - x_0\|^3},$$

and

$$\ddot{x}_j = m_0 \frac{x_0 - x_j}{\|x_0 - x_j\|^3} + \epsilon \sum_{k \geq 1, k \neq j} m_k \frac{x_k - x_j}{\|x_k - x_j\|^3} \quad (j = 1, \dots, n).$$

We think of the body 0 as the Sun and the other bodies as  $n$  planets revolving around the Sun. In our Solar System, the mass of Jupiter is 1/1000 that of the Sun, which justifies that we consider small values of  $\epsilon$ . The equations have a limit when  $\epsilon \rightarrow 0$ , for which the Sun is still (in an appropriate frame of reference) and each planet undergoes the only attraction of the Sun. If their energies are negative, planets describe Keplerian ellipses, with some fixed semi major axes and excentricities. As a whole, the system is quasiperiodic with  $n$  frequencies. For a generic Hamiltonian system with  $3n$  degrees of freedom, one would expect  $3n$  frequencies; due to this dynamical degeneracy of the Newtonian potential, we are dealing with a singular perturbation problem.

In 1963, V. Arnold [2] published the following remarkable result.

**Theorem 1.** *For every  $m_0, m_1, \dots, m_n > 0$  and for every  $0 < a_1 < \dots < a_n$  there exists  $\epsilon_0 > 0$  such that for every  $0 < \epsilon < \epsilon_0$ , in the phase space in the neighborhood of circular and coplanar Keplerian motions with semi major axes  $a_1, \dots, a_n$ , there is a subset of positive Lebesgue measure of initial conditions leading to quasiperiodic motions with  $3n - 1$  frequencies.*

The proof of this theorem is rendered difficult by the multitudinous degeneracies of the planetary system. Arnold’s initial proof does not fully describe these degeneracies. Actually Arnold’s strategy fails, in its straightest form, due to a resonance he had not foreseen (Herman’s resonance). In

1998, in a series of lectures M. Herman sketched a complete and more conceptual proof of this theorem, showing some weak non-degeneracy property of the planetary problem, and then concluding by calling upon a theorem of Diophantine approximation of Arnold-Pyartli [7]. Later, Chierchia and Pinzari strengthened the result qualitatively and quantitatively with a slightly different proof closer to Arnold initial strategy, showing in particular the strong non-degeneracy property of the planetary problem [5, 6]. We now review some ideas of these proofs.

**1 Hamiltonian** Newton's equations are equivalent to Hamilton's equations (Cauchy, 1831)

$$\begin{cases} \dot{x}_0 = \epsilon \partial_{y_0} H \\ \dot{y}_0 = -\epsilon \partial_{x_0} H \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_j = \partial_{y_j} H \\ \dot{y}_j = -\partial_{x_j} H \end{cases}$$

if the linear momenta are  $y_0, \epsilon y_1, \dots, \epsilon y_n$  and the Hamiltonian

$$\epsilon H = \frac{1}{2} \frac{\|y_0\|^2}{m_0} + \epsilon \sum_j \left( \frac{\|y_j\|^2}{2m_j} - \frac{m_0 m_j}{\|x_j - x_0\|} \right) - \epsilon^2 \sum_{j < k} \frac{m_j m_k}{\|x_j - x_k\|}$$

(planets's indices  $j, k$  vary from 1 to  $n$ ).

**2 Reduction by translations** Switch to the symplectic heliocentric coordinates:

$$\begin{cases} X_0 = x_0 \\ Y_0 = y_0 + \epsilon y_1 + \epsilon y_2 + \dots + \epsilon y_n \end{cases} \quad \text{and} \quad \begin{cases} X_j = x_j - x_0 \\ Y_j = y_j. \end{cases}$$

The conservation of the total linear momentum  $Y_0$  and the invariance by translation of Newton's equations allow us to focus on the subspace  $Y_0 = 0$  without loss of generality, and to ignore the variable  $X_0$ . The equations now read

$$\dot{X}_j = \partial_{Y_j} H, \quad \dot{Y}_j = -\partial_{X_j} H,$$

with

$$H = \sum_j \left( \frac{\|Y_j\|^2}{2\mu_j} - \frac{\mu_j M_j}{\|X_j\|} \right) + \epsilon \sum_{j < k} \left( -\frac{m_j m_k}{\|X_j - X_k\|} + \frac{Y_j \cdot Y_k}{m_0} \right);$$

the 'fictitious' masses  $\mu_j \sim m_j$  and  $M_j \sim m_0$  are functions of the  $m_j$ 's and of  $\epsilon$ , defined by  $\frac{1}{\epsilon \mu_j} = \frac{1}{m_0} + \frac{1}{\epsilon m_j}$  and  $M_j = m_0 + \epsilon m_j$ .

**3 The Keplerian part and the perturbing function** The Hamiltonian splits into  $H = H_{Kep} + \epsilon H_{per}$ , the sum of a *Keplerian Hamiltonian* and an  $\epsilon$ -small *perturbing function*.

Let  $(\lambda_j, \Lambda_j, \xi_j, \eta_j, p_j, q_j) \in \mathbb{T}^1 \times ]0, +\infty[ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  be the Poincaré coordinates of the  $j$ -th planet. Those coordinates are analytic and symplectic on a neighborhood of the union of circular, direct, horizontal (with respect to some given plane in space) Keplerian ellipses:

- the angle  $\lambda_j$  parameterizes the ellipse proportionally to the area swept by the position vector,
- $\Lambda_j = \mu_j \sqrt{M_j a_j}$ ,
- $\xi_j$  and  $\eta_j$  determine the eccentricity and orientation of the ellipse within its plane,
- $p_j$  and  $q_j$  determine the position of the plane of the ellipse.

Those coordinates straighten the (degenerate) Keplerian dynamics, since  $H_{Kep}$  now only depends on  $\Lambda_j$ 's:

$$H_{Kep} = \sum_j -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2}.$$

Let

$$\nu_j = \frac{\partial H_{Kep}}{\partial \Lambda_j} = \frac{\mu_j^3 M_j^2}{\Lambda_j^3} = \frac{\sqrt{M_j}}{a_j^{3/2}}$$

be the so-called *mean motions* (a weird, old name for the Keplerian frequencies); Kepler's third law follows from this expression of  $\nu_j$ .

**4 The averaged Hamiltonian** From now on, restrict to the open set, diffeomorphic to  $\mathbb{T}^n \times \mathbb{R}^{5n}$ , over which Keplerian ellipses do not meet. Up to renumbering, we can assume that  $0 < a_1 < \dots < a_n$ . Away from the boundary and in a neighborhood of circular Keplerian ellipses, the perturbing function is uniformly  $\epsilon$ -small. Thus a change of coordinates  $\epsilon$ -close to the identity transforms  $H$  into the *averaged Hamiltonian*

$$H_{Kep} + \epsilon \langle H_{per} \rangle = H_{Kep} + \epsilon \int_{\mathbb{T}^n} H_{per} \frac{d\lambda_1 \dots d\lambda_n}{(2\pi)^n},$$

up to terms of order 2 in  $\epsilon$ , along the Cantor set of Diophantine Keplerian frequencies. (Technically, this means that the new Hamiltonian equals the above expression, plus a term which is  $C^\infty$ -flat on the Cantor set, which plays no role and which we will ignore here.)

For the averaged Hamiltonian, the momenta  $\Lambda_j$  are first integrals (this is the first stability theorem of Lagrange and Laplace). It descends to the quotient by the Keplerian action of  $\mathbb{T}^n$  and induces a Hamiltonian system on the space, diffeomorphic to  $\mathbb{R}^{4n} = \{(\xi_j, \eta_j, p_j, q_j)_{j \in \{1, \dots, n\}}\}$ , of Keplerian tori with fixed semi major axes. This Hamiltonian, up to a constant, is  $\langle H_{per} \rangle$ ; it

is called the *secular Hamiltonian* and its phase space the *secular space*. This system describes the slow variations of excentricity and orientation in space of the Keplerian ellipses of the planets, under the influence of mutual attraction, at the first order in  $\epsilon$ , outside resonances in mean motions. Contrary to its analogue in the Lunar problem (see [8]), it seems not integrable.

**5 The elliptic secular singularity** By symmetry, the origin  $\xi = \eta = p = q = 0$  of the secular space is an equilibrium point. It turns out to be an elliptic critical point of the averaged Hamiltonian (this is the second stability theorem of Laplace), having the following remarkable expansion:

$$\langle H_{per} \rangle = C_0(m, a) + Q_h \cdot \xi^2 + Q_h \cdot \eta^2 + Q_v \cdot p^2 + Q_v \cdot q^2 + O(4),$$

where the “horizontal” and “vertical” quadratic forms  $Q_h$  and  $Q_v$  are of the form

$$\begin{cases} Q_h \cdot \xi^2 &= \sum_{1 \leq j < k \leq n} m_j m_k \left( C_1(a_j, a_k) \left( \frac{\xi_j^2}{\Lambda_j} + \frac{\xi_k^2}{\Lambda_k} \right) + 2C_2(a_j, a_k) \frac{\xi_j \xi_k}{\sqrt{\Lambda_j \Lambda_k}} \right) \\ Q_v \cdot p^2 &= \sum_{1 \leq j < k \leq n} -m_j m_k C_1(a_j, a_k) \left( \frac{p_j}{\sqrt{\Lambda_j}} - \frac{p_k}{\sqrt{\Lambda_k}} \right)^2; \end{cases}$$

coefficients  $C_j$  are real analytic with respect to the semi major axes, and can be expressed in terms of the Laplace coefficients.

**6 The secular frequencies** Let  $\rho_h$  and  $\rho_v$  be orthogonal diagonalizing transformations of  $Q_h$  and  $Q_v$  (depending analytically on the masses and semi major axes):

$$\rho_h^* Q_h = \sum_j \sigma_j d\xi^2 \quad \text{and} \quad \rho_v^* Q_v = \sum_j \varsigma_j dp^2.$$

In the full phase space, the map

$$\rho : (\xi, \eta, p, q) \mapsto (\rho_h \cdot \xi, \rho_h \cdot \eta, \rho_v \cdot p, \rho_v \cdot q)$$

is symplectic and, the problem thus boils down to studying a Hamiltonian of the form

$$\begin{aligned} H_{Kep}(\Lambda) + \epsilon \sum_j (\sigma_j(\Lambda)(\xi_j^2 + \eta_j^2) + \varsigma_j(\Lambda)(p_j^2 + q_j^2)) \\ + \epsilon O_4(\xi, \eta, p, q) + O_2(\epsilon^2), \end{aligned}$$

where remainders do not depend on  $\lambda$ . The Hamiltonian obtained by neglecting those remainders is integrable and its integral curves (outside the elliptic singularity) are quasiperiodic, with the  $3n$  frequencies given by

$$\alpha^o := (\nu_1, \dots, \nu_n, \epsilon\sigma_1, \dots, \epsilon\sigma_n, \epsilon\varsigma_1, \dots, \epsilon\varsigma_n).$$

The arithmetic properties of the frequency vector  $\alpha^o$  play a deciding role in the existence of higher order normal forms;  $\alpha^o$  depends on the masses and the semi major axes, and can also be thought of as a constant function of the actions  $\xi_j^2 + \eta_j^2$  and  $p_j^2 + q_j^2$  (when one moves away from the elliptic secular singularity).

**7 Properties of the frequency vector** KAM theory asserts that there is a perturbed frequency vector  $\alpha$ , not explicitly known but tending to  $\alpha^o$  when  $\epsilon$ ,  $\xi_j^2 + \eta_j^2$  and  $p_j^2 + q_j^2$  tend to zero, such that whenever  $\alpha$  is Diophantine there is a corresponding invariant torus.

It is customary in this theory to measure the abundance of invariant tori given by the KAM theorem, with fixed masses. J. Moser justified this habit with the argument that one cannot change the masses of the planets. We will let the reader decide for himself whether it is easier to ‘change’ semi major axes.

— If  $\alpha$  is seen as a function of  $a = (a_1, \dots, a_n)$  alone (implying that we look for tori very close to circular coplanar Keplerian ellipses), the best we can hope for is the weak non-degeneracy property of Arnold-Pyartli i.e., the local image of  $\alpha$  does not lie in a hyperplane. In this case, the theory of Diophantine approximations shows that the set of parameters  $a$  such that the frequency vector is Diophantine has positive Lebesgue measure.

— If  $\alpha$  is seen as a function of both  $a$  and the actions  $\xi_j^2 + \eta_j^2$ ’s and  $p_j^2 + q_j^2$ ’s, one can hope for the strong non-degeneracy property of Kolmogorov i.e., the map  $\alpha$  is a local diffeomorphism, thus obviously reaching Diophantine vectors for a set of parameters of positive measure.

There are difficulties in both cases, which are removed when one takes advantage of the invariance of the system by rotations. This symmetry allows us to decrease the number of degrees of freedom by two units and get rid of annoying resonances.

**8 Weak non-degeneracy** In the plane ( $p = q = 0$ ), the application of KAM theory is quite straightforward, due to the following fact.

**Proposition 2.** *Outside an analytic proper subset of values of  $a = (a_1, \dots, a_n)$ , the vector  $(\nu_1, \dots, \nu_n, \epsilon\sigma_1, \dots, \epsilon\sigma_n)$  locally defines an analytic function of  $a$ , whose image is not contained in any vector hyperplane.*

This means that no resonance is identically verified and, by Pyartli’s theorem [13], that the perturbed frequency vector passes through Diophantine vectors in positive measure. Arnold’s theorem in the plane follows. The proof of the proposition goes along these lines.

For  $n = 2$  planets, there is an explicit asymptotics of  $\alpha$  in the ‘well-spaced regime’ where  $a_1 \ll a_2$ . Then the conclusion of the proposition is readily checked. Complexifying the semi major axes and using the analytic

continuation of  $Q_h$ , one can deduce the conclusion of the proposition with  $n = 2$ . In the case of  $n + 1$  planets, in the well-spaced regime the spectrum of  $Q_h$  splits into a very small eigenvalue and the spectrum corresponding to  $n$  planets. By induction over the number  $n$  of planets the conclusion follows.

In space, the situation is more intricate.

**Proposition 3.** *Outside an analytic proper subset of values of  $a$ , the frequency vector  $\alpha^o$  locally defines an analytic function of  $a$ , whose image is contained in the codimension-2 space of  $\mathbb{R}^{3n}$  of equations*

$$\varsigma_n = 0, \quad \sum_j (\sigma_j + \varsigma_j) = 0,$$

*(up to reordering of the  $\varsigma_j$ 's) but is contained in no plane of larger codimension.*

The proof of the proposition goes along the same lines as for the plane problem.

Unsurprisingly, the first resonance is due to the invariance of the system by rotations about any horizontal axis (hence the linearized secular vector field has two vanishing eigenvalues), so that one of the spatial secular frequencies, say  $\varsigma_n$ , vanishes. Thus this resonance disappears when one restricts to a fixed direction, say vertical, of the angular momentum (a codimension-2, hence  $(6n - 2)$ -dimensional, symplectic submanifold).

The second resonance is mysterious and does not seem to be associated with any symmetry. Although it was known to astronomers in particular cases ( $n = 2$ ), it was discovered in its full generality by Herman. It happens to disappear when one completes the reduction by rotations, by fixing the angular momentum vector and quotienting by rotations around the fixed direction of the angular momentum. For  $n = 2$  planets, this is checked by carrying out the classical reduction of the node of Jacobi; Arnold did so without noticing that he did get rid of a resonance in the process.

The difficulty when  $n \geq 3$  is that we do not know of simple coordinates adapted to the reduction by rotations. Arnold thus suggested to merely fix the direction of the angular momentum, not being aware of Herman's resonance. In order to check without much computation that Herman's resonance does not exist at the fully reduced level, one can use a trick (also used by Poincaré in order to find some periodic orbits), consisting in adding to the Hamiltonian a term in  $\delta \|C\|$ , where  $\delta$  is small real number, and  $\|C\| = C_z$  is the length of the angular momentum. This amounts to switching to a rotating frame of coordinates. The KAM theorem applies to show the existence of Diophantine Lagrangian tori of dimension  $3n - 2$ , invariant for the modified Hamiltonian, for a positive measure of values of  $\delta$  (one value of  $\delta$  would suffice). Then, by an argument of Lagrangian intersection, these tori must be invariant by the flow of the initial Hamiltonian. Adding back

the frequency corresponding to the rigid rotation of the system, one gets tori of dimension  $3n - 1$ , which are sometimes ergodic, and sometimes foliated in ergodic invariant tori of dimension  $3n - 2$ . This proves Arnold's theorem.

**9 Strong non-degeneracy** Chierchia and Pinzari have strengthened and completed the previous result by investigating the part of the secular Hamiltonian which is quartic with respect to the secular Poincaré coordinates  $\xi, \eta, p, q$ , as in Arnold's original strategy. As above, the invariance by rotations about a horizontal axis prevents the non-reduced secular system from being non-degenerate, this time in the sense of Kolmogorov. But this obstruction vanishes in restriction to the codimension-2 submanifold obtained by fixing the direction of the angular momentum.

It happens that Herman's resonance does not create any resonant term at the fourth order (nor until the 10-th order [9]). So it does not prevent to compute the torsion in the partially reduced system (fixed direction of the angular momentum, but norm of the angular momentum not fixed and no quotient by rotation around the angular momentum). This partial reduction by rotations, which already played a key role above for proving the weak non-degeneracy condition, can be understood as follows. Topologically, the group  $SO_3$  is an  $S^1$ -bundle over  $S^2$ . The action of the maximal torus  $S^1 = SO(2)$  carries all the non-trivial dynamical information, while all directions of the angular momentum are equivalent to each other.

The fourth order terms are formidable to compute. Chierchia-Pinzari use some regularized version of the Deprit coordinates. Those coordinates do not reduce to coordinates per planet. The Deprit coordinates were independently rediscovered by G. Pinzari during her PhD [11], before she realized they matched Deprit's coordinates. Interestingly, Deprit believed that nobody would ever make anything useful with these coordinates. Pinzari managed to compute asymptotics, as before in the well-spaced regime, of the 4-th order terms of the averaged Hamiltonian. Chierchia-Pinzari can conclude triumphantly, probably as much as possible in the spirit of what Arnold had in mind in 1963:

**Theorem 4.** *The torsion of the planetary problem at the elliptic singularity of the averaged system is non-degenerate, both in restriction to the submanifold obtained by fixing the direction of the angular momentum, and in the system fully reduced by rotations.*

Hence KAM theory can be applied to the planetary problem, with full strength. Chierchia-Pinzari's theorem gives some additional information on the secular system and allows Chierchia-Pinzari to give better measure estimates than what the weak non-degeneracy would give. (Un)fortunately, the torsion is indefinite, which prevents from using variational methods to (easily?) find invariant sets from Mather theory...

## A Poincaré coordinates

For the Kepler Hamiltonian

$$K = \frac{\|p\|^2}{2\mu} - \frac{\mu M}{\|q\|}, \quad (q, p) \in \mathbb{R}_*^3 \times \mathbb{R}^3,$$

we will first define some symplectic action-angle coordinates, attributed to Delaunay, which blow up circular or horizontal Keplerian motions; blowing down the singularity will yield the symplectic analytic Poincaré coordinates. For elementary facts about  $K$ , we refer to [3]. For other ways to introduce those coordinates, see [15, Chap. VII], or [4, 10, 12, 14].

Notations in the plane Kepler problem with negative energy:

$v$  = true anomaly = argument of  $q$ , measured from the pericenter

$a$  = semi major axis

$g$  = argument of the pericenter from  $Ox$

$\epsilon$  = eccentricity.

Restrict to the set  $\{(v, a, g, \epsilon), 0 < \epsilon < 1\} \equiv \mathbb{T}^2 \times \mathbb{R}^2$  of non-degenerate, non-circular, elliptical, Keplerian motions. Define coordinate  $t$  as the time from the pericenter; it is defined modulo the period  $T$  of the orbit. Define  $\ell$ , the *mean anomaly*, as the angle obtained by rescaling time:  $\ell := 2\pi t/T \pmod{2\pi}$ . Now, if we want an action coordinate  $L(K)$  conjugate to  $\ell$ :  $dt \wedge dK = d\ell \wedge dL$ , we see that

$$L'(K) = \frac{1}{\dot{\ell}} = \frac{T}{2\pi} = \frac{a^{3/2}}{\sqrt{M}} = \frac{\mu^{3/2}M}{(-2K)^{3/2}}.$$

Conventionally choosing  $L = 0$  at infinity where  $a = +\infty$ , we get

$$L = \frac{\mu^{3/2}M}{\sqrt{-2K}} = \mu\sqrt{Ma} \quad \left( \text{so that } K = -\frac{\mu^3 M^2}{2L^2} \right).$$

We now wish to define coordinates on the space of non-circular, non-degenerate, prograde Keplerian ellipses in the plane with fixed  $L$ . The *angular momentum*

$$G := \mu\sqrt{Ma(1-\epsilon^2)} = L\sqrt{1-\epsilon^2},$$

is a first integral of  $K$  and thus descends to the space of Keplerian orbits. Its Hamiltonian flow acts by  $2\pi$ -periodic rotations around the origin. Define  $g$  as the angle, modulo  $2\pi$ , measuring time along  $X_G$ -orbits, and vanishing when the pericenter meets the  $Ox$ -semi-axis.

The coordinates which  $(\ell, L, g, G)$  define are symplectic:

- $\{\ell, L\} = \{g, G\} = 1$  by definition.
- $\{L, g\} = \{L, G\} = 0$  because  $g$  and  $G$  are first integrals of  $K(L)$ .
- $\{\ell, G\} = 0$  because the flow of  $G$  rotates the Keplerian ellipse without revolving the body along the ellipse.



–  $\{\ell, g\} = 0$ . Due to the Jacobi identity,  $\{L, \{\ell, g\}\} = \{G, \{\ell, g\}\} = 0$ . Hence it suffices to show that  $\{\ell, g\} = 0$  in restriction to the section  $\{\ell = g = 0 \pmod{\pi}\}$  of the  $L$ - and  $G$ -flows. We may thus assume that the body is on the major axis and that the major axis itself is the  $x$ -axis. But then the partial derivatives of  $\ell$  and  $g$  with respect of  $x$  or  $p_y$  are zero, and

$$\{\ell, g\} = \underbrace{\frac{\partial \ell}{\partial x}}_{=0} \frac{\partial g}{\partial p_x} - \frac{\partial \ell}{\partial p_x} \underbrace{\frac{\partial g}{\partial x}}_{=0} + \frac{\partial \ell}{\partial y} \underbrace{\frac{\partial g}{\partial p_y}}_{=0} - \underbrace{\frac{\partial \ell}{\partial p_y}}_{=0} \frac{\partial g}{\partial y} = 0.$$

In the 3-dimensional Kepler problem, choose  $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$  as a reference plane, called *horizontal*. Temporarily restrict to non-horizontal, non-circular, non-degenerate, prograde elliptic Keplerian motions.

Let  $\vec{C} = q \times p$  be the *angular momentum vector* and  $\Theta$  be its projection on the vertical axis. The flow of  $X_\Theta$  consists of  $2\pi$ -periodic rotations in the horizontal plane (diagonally for positions and impulsions), leaving the horizontal 4-plane invariant.

Each Keplerian oriented plane meets the horizontal plane along a half axis, the *ascending line of the node*. Let  $\theta$  be the angle measuring time along  $X_\Theta$ -orbits, vanishing when the line of the node is the  $Ox$ -semi-axis.

The so-defined coordinates  $(\ell, L, g, G, \theta, \Theta)$  are symplectic:

– Poisson brackets with  $L, G$  and  $\Theta$  are what they should : 0, except  $\{\ell, L\} = \{g, G\} = \{\theta, \Theta\} = 1$  (we know the flows of  $L, G$  and  $\Theta$ ).

– The three Poisson brackets between angles can be checked to vanish as above in the plane. Indeed, on the submanifold  $\{\ell = g = \theta = 0 \pmod{\pi}\}$ , the partial derivatives of any of these angles with respect to  $x, p_y$  or  $p_z$  vanish.

Now, define the Poincaré coordinates  $(\lambda, \Lambda, \zeta, z)$  by the following formulas (several sign conventions exist):

$$\begin{cases} \lambda = \ell + g + \theta \\ \Lambda = L \\ \zeta = \sqrt{2(L - G)} e^{-ig} \\ z = \sqrt{2(G - \Theta)} e^{-i\theta} \end{cases}$$

Knowing that the Delaunay coordinates are symplectic, it is straightforward to check that the Poincaré coordinates are symplectic too. From the above formulas, one checks that the Poincaré coordinates extend to continuous coordinates at direct circular coplanar motions ( $\zeta = z = 0$ ), since

$$\begin{cases} \zeta = \sqrt{\Lambda/2} \epsilon (1 + O(\epsilon^2)) e^{-ig} \\ z = \sqrt{\Lambda/2} \iota (1 + O(\epsilon^2) + O(\iota^2)) e^{-i\theta}, \quad \cos \iota = \frac{\Theta}{G} \end{cases}$$

(in particular,  $g$  and  $\iota$  need not be defined at the singularity). In fact, their extension is analytic, as one can see by expressing the coordinates as explicit

analytic functions of analytic first integrals; see [1] for an elegant choice of first integrals.

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## References

- [1] K. Abdullah and A. Albouy. On a strange resonance noticed by M. Herman. *Regul. Chaotic Dyn.*, 6(4):421–432, 2001.
- [2] V. I. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Uspehi Mat. Nauk*, 18(6 (114)):91–192, 1963.
- [3] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical aspects of classical and celestial mechanics*, volume 3 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, third edition, 2006. Translated from the Russian original by E. Khukhro.
- [4] A. Chenciner. Intégration du problème de Kepler par la méthode de Hamilton-Jacobi. Technical report, Bureau des Longitudes, 1989.
- [5] L. Chierchia and G. Pinzari. Planetary Birkhoff normal forms. *J. Mod. Dyn.*, 5(4):623–664, 2011.
- [6] L. Chierchia and G. Pinzari. The planetary  $N$ -body problem: symplectic foliation, reductions and invariant tori. *Invent. Math.*, 186(1):1–77, 2011.
- [7] J. Féjoz. Démonstration du ‘théorème d’Arnold’ sur la stabilité du système planétaire (d’après Herman). *Ergodic Theory Dynam. Systems*, 24(5):1521–1582, 2004.
- [8] M. L. Lidov and S. L. Ziglin. Non-restricted double-averaged three body problem in Hill’s case. *Celestial Mech.*, 13(4):471–489, 1976.
- [9] F. Malige, P. Robutel, and J. Laskar. Partial reduction in the  $n$ -body planetary problem using the angular momentum integral. *Celestial Mechanics and Dynamical Astronomy*, 84:283–316, 2002. 10.1023/A:1020392219443.
- [10] J. Moser and E. J. Zehnder. *Notes on dynamical systems*, volume 12 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2005.
- [11] G. Pinzari. *On the Kolmogorov Set for Many-Body Problems*. PhD thesis, Università di Roma Tre, 2009.

- [12] H. Poincaré. *Leçons de mécanique céleste*. Gauthier-Villars, 1905.
- [13] A. S. Pyartli. Diophantine approximations of submanifolds of a Euclidean space. *Funkcional. Anal. i Priložen.*, 3(4):59–62, 1969.
- [14] E. L. Stiefel and G. Scheifele. *Linear and regular celestial mechanics. Perturbed two-body motion, numerical methods, canonical theory*. Springer-Verlag, New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 174.
- [15] F. Tisserand. *Traité de mécanique céleste*. Gauthier-Villars (Paris), 1896.